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FUZZY VECTOR OBJECTIVE OPTIMIZATION ALGORITHM FOR IMAGE RECONSTRUCTION FROM INCOMPLETE PROJECTIONS

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This paper deals with the problem of image reconstruction from incomplete projections. A novel fuzzy vector objective optimization model is developed by integrating the fuzzy set theory and vector objective optimization (multi-objective decision-making). The objective function is expressed as a membership function, and the minimum operator is taken as a fuzzy operator. Furthermore, a novel iterative method is proposed to resolve the fuzzy optimization problem. The images reconstructed from simulated noise projections and real projections obtained from an industrial scanner show that the new algorithm can provide higher resolution and better smoothness than the images reconstructed by the transformation method and the conventional iterative method, so it is more feasible for image reconstruction from incomplete projections.

Keywords: Image reconstruction; fuzzy vector objective optimization; incomplete projections; iterative method.

1. Introduction

Image reconstruction from projections is widely used in observation of detailed structure inside an object and it can be classified into two categories: (1) the transformation method based on Radon transformation, e.g. Convolution Back Projection (CBP);[1] (2) the iterative method based on algebra, e.g. Algebraic Reconstruction Techniques (ART).[2–5] The transformation method is efficient with sufficient and exact projections, whereas the iterative method is more preferred for image reconstruction using incomplete or/and unevenly distributed projections.

An optimization method for iterative image reconstruction was first developed by Kashyap and Mittal,[6] in which an appropriate objective function is selected and an effective algorithm is developed to optimize the function. It is termed single objective optimization image reconstruction,[3,7–11] which focuses on some specification in regions being imaged, such as degree of fitting. But it may not account for other specifications such as smoothness. Thus qualities of the images reconstructed by the single objective optimization method may not be satisfactory. For incomplete projections, the information available for imaging is often not sufficient. And it is desirable that the relativities among the incomplete projections should be adequately utilized. One of the solutions is that multiple specifications are considered simultaneously for reconstructing the region of interest (ROI), and these specifications are equilibrated, so as to obtain desirable reconstruction of the image. This is the motivation of vector objective optimization for image reconstruction. With the aid of fast iterative algorithms,[12,13] the vector objective optimization method for image reconstruction, especially for imaging from incomplete projections, has been well known.[14–20]

Although the vector objective optimization problem differs from single objective optimization problem only in the plurality of objective functions, it should be noted that multiple objectives are often incommensurable and conflict between each other. In the vector objective optimization, the concept of Pareto optimum solution or efficient solution, which is superior or inferior with respect to the various objectives, is introduced instead of the optimality concept of single objective optimization. Decisions with Pareto optimum solution or efficient solution are, however, not uniquely determined, and the final decision must be selected from the set of Pareto optimum solutions or efficient solutions. Thus, a key issue of the vector objective optimization method is to obtain a compromise or satisfactory solution of a decision-maker (i.e. best trade-off solution) as the final solution, from either the Pareto optimum solution or the efficient solution set.

Recently, some researchers studied the vector objective optimization for image reconstruction and proposed some optimization techniques.[14–20] For example, vector objective functions can be incorporated into a single objective function as a weighted sum. However, the improvement in image quality and reduction in time consumption are difficult to achieve mainly due to two reasons. First, it is difficult to equilibrate multiple objectives. In a regularized or penalized approach, selection of the weighted coefficient, which greatly influences the quality of the reconstructed images, is mostly decided subjectively. For one region being imaged from incomplete projections, different decision-makers have different partialities without regularity. The second reason is that the vector in the image reconstruction is of high dimension which demands a great deal of the iterative processing time.

The fuzzy set theory has been widely used in various areas in recent years and the fuzzy vector objective optimization is a relatively new technique. Considering that the noise in projections is stochastic and uncertain information and that the equilibrium of various objectives in the vector objective optimization can be formulated into a fuzzy problem, this paper presents a novel algorithm for image reconstruction using fuzzy vector objective optimization. The membership functions are used to depict objective functions, and the fuzzy operator is adopted to evaluate the whole satisfactory index of the objectives corresponding to their respective optimality. Based on these, a mathematical algorithm for the image reconstruction is developed. The images reconstructed from simulated noise projections and real projections obtained from an industrial scanner demonstrate that the new method revealed higher resolution and better smoothness than the images reconstructed by the transformation method and the conventional iterative method. It is shown that the proposed method has advantages in image reconstruction using incomplete projections.

2. Image and Projection Representation

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ denotes an n-dimensional reconstructed image vector, where x_j is the density value of the jth pixel in the reconstructed image. Let y_i denotes the ray integral measured with the ith ray, and $\mathbf{y} = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$ denotes the projection vector. Let $\mathbf{A} = (a_{ij})_{m\times n}$ be $m \times n$ projections matrix, where a_{ij} is a weight factor representing the contribution of the jth pixel to the ith ray integral. The image reconstruction from projections can be expressed as

$$\begin{array}{ll} \displaystyle\sum_{j=1}^{\mathrm{n}} a_{ij}x_j = y_i & i = 1, 2, \ldots, m \\ \mathbf{A}_i\mathbf{x} = \mathrm{y}_i & i = 1, 2, \ldots, m \end{array}, \tag{1}$$

where $\mathbf{A}_i = (\mathrm{a}_{i1}, \mathrm{a}_{i2}, \ldots, \mathrm{a}_{i\mathrm{n}})$, and $\mathbf{A}_i\mathbf{x}$ can be considered as a reprojection of the ith ray passing through every pixel of the reconstructed image.

Since real projections may contain noise, a noise vector $\mathbf{e} = (e_1, e_2, \ldots, e_m)^{\mathrm{T}}$ is introduced to the projection model. Assume that e_i is represented independently of $(0, \sigma_{\mathrm{i}}^2)$ Gaussian distribution. The model of image reconstruction from projections may be expressed as

$$\mathbf{A}_i\mathbf{x} + \mathrm{e}_i = \mathrm{y}_i \quad i = 1, 2, \ldots, m. \tag{2}$$

As a result, the reconstruction problem is formulated as an inverse problem, which estimates the image vector $\mathbf{x}$ from the measured vector $\mathbf{y}$ and the geometrical parameter of projection $\mathbf{A}$.

3. Model of Vector Objective Optimization for Image Reconstruction

The model of vector objective optimization for image reconstruction from projections is formulated as

$$\begin{cases} \min & \mathbf{F}(\mathbf{x}) = (\mathrm{f}_1(\mathbf{x}), \quad \mathrm{f}_2(\mathbf{x}), \ldots, \mathrm{f}_L(\mathbf{x})) \\ \text{s.t.} & \mathbf{A}_i\mathbf{x} + e_i = y_i \quad\quad i = 1, 2, \ldots, m \end{cases}, \tag{3}$$

where $\mathrm{f}_l(\mathbf{x})$ is an objective function, and L is the total number of objective functions. There are a number of single objective optimization methods to the solution of the reconstruction problem, including the norm minimization,[7] the least-squares,[3] the general quadratic optimum,[4] the Bayesian,[8,9] the maximum-likelihood,[10] the maximum entropy,[11] and the minimum variance.[12]

For image reconstruction from incomplete projections, a reconstructed image may be a solution for the following optimization specifications:

(1) Minimal error between real projections and reprojections through reconstructed image. It makes the reconstructed image vector $\mathbf{x}$ to be as close to original image as possible.

$$\min \mathrm{f}_1(\mathbf{x}) = ||\mathbf{A}\mathbf{x} - \mathbf{y}||^2. \tag{4}$$

(2) Maximal entropy of the image vector $\mathbf{x}$. It assures that the reconstructed image has the highest global smoothness.[11]

$$\min \mathrm{f}_2(\mathbf{x}) = \sum_{j=1}^{n} x_j \ln x_j. \tag{5}$$

(3) Minimal norm of the image vector $\mathbf{x}$. It assures highest local smoothness and lowest peak level in the reconstructed image,[19]

$$\min \mathrm{f}_3(\mathbf{x}) = \frac{1}{2}(\mathbf{x}^{\mathrm{T}}\mathbf{A} \cdot \mathbf{x}), \tag{6}$$

where $\mathbf{A}$ is a covariance matrix which has low-pass filter property.[4]

There are errors during real projection. They are introduced into the reconstruction model to be a constraint condition[4] as:

$$\mathrm{h}(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m} \frac{(\mathbf{A}_i\mathbf{x} - y_i)^2}{\sigma_i^2}. \tag{7}$$

The following equation is introduced based on the central limit in probability theory:

$$\lim_{m\to\infty} \quad \frac{1}{m}\sum_{i=1}^{m} \frac{(\mathbf{A}_i\mathbf{x} - y_i)^2}{\sigma_i^2} = 1. \tag{8}$$

In general, m (the total number of ray projection) is large and the order of magnitude of m is up to 10^5. Therefore, Eq. (7) can be approximately presented as follows:

$$\mathrm{h}(\mathbf{x}) = \frac{m}{2}. \tag{9}$$

The constraint $h(\mathbf{x})$ determines the set of feasible image vector $\mathbf{x}$.[19] Thus, the model of vector objective optimization for image reconstruction is obtained:

$$\begin{aligned} &\min \mathbf{f}(\mathbf{x}) = (\mathrm{f}_1(\mathbf{x}), \mathrm{f}_2(\mathbf{x}), \mathrm{f}_3(\mathbf{x})) \\ &\text{s.t.}\;\; \mathrm{h}(\mathbf{x}) = \frac{m}{2} \end{aligned} . \tag{10}$$

A common solution to this problem is by using the utility function method.[21] The utility function is defined by decision-maker according to the preference information to transform the vector objective optimization problem to the single objective optimization problem.

4. Fuzzy Vector Objective Optimization (FVOO) Algorithm for Image Reconstruction

Besides the utility function method, other methods, such as objective programming, interactive decision-making and fuzzy programming, can be employed to obtain a trade-off solution of decision-making for the vector objective optimization problem.

The FVOO algorithm is developed in this paper for the following reasons: (1) A large amount of stochastic and uncertain fuzzy information exists during image reconstruction, such as noise in real projections. (2) It is difficult to optimize various objective functions simultaneously, which sometimes are mutually conflictive and noncommensurable. When one objective is optimized, the others degrade. Thus the equilibrium of various objectives generally is a fuzzy problem. (3) The imprecision and fuzziness are inherent in human decision-making. In the case of such vagueness, the vector objective optimization image reconstruction under fuzziness was superior and preferred. (4) Compared with conventional trade-off methods, the FVOO method does not need normalized coefficient and weight coefficient.

To develop FVOO, several symbols are introduced first. The membership function[22] of $\tilde{A}$ is denoted as $\mu_{\tilde{A}}$. The satisfactory index,[23] which the decision-making x arrives at relative to goal $\tilde{G}$, is denoted as $\mu_{\tilde{G}}$. Other concepts of fuzzy vector objective optimization can be seen in references.[22,23]

There are some issues to be considered in the development of FVOO algorithm for image reconstruction: (1) A proper membership function is selected to transform the general objective function to a fuzzy goal function. (2) Various fuzzy goals are integrated into one by using one or more fuzzy operators. (3) A mathematical model of fuzzy vector objective optimization should be established. (4) A reconstruction procedure is developed.

In this paper, the fuzzy goal is depicted by linear membership function:

$$\mu_i(\mathrm{f}_i(\mathbf{x})) = \frac{\mathrm{f}_i^{\max}(\mathbf{x}) - \mathrm{f}_i(\mathbf{x})}{\mathrm{f}_i^{\max}(\mathbf{x}) - \mathrm{f}_i^{\min}(\mathbf{x})}, \tag{11}$$

where $\mathrm{f}_i^{\max}(\mathbf{x}) = \max_{x \in X} \mathrm{f}_i(\mathbf{x})$, $\mathrm{f}_i^{\min}(\mathbf{x}) = \min_{x \in X} \mathrm{f}_i(\mathbf{x})$ under the given constraints. The expression is straightforward and convenient. The solution is for the extremum of objective function but it increases computational complexity. The maxima and minima of the objective functions are directly given by their estimate. Corresponding to the three objective functions (4)∼(6), their extrema are, respectively (see Appendix),

$$\begin{aligned} &\mathrm{f}_1^{\max} = \frac{m}{2}, \quad \mathrm{f}_1^{\min} = 0 \\ &\mathrm{f}_2^{\max} = 0, \quad \mathrm{f}_2^{\min} = \ln\frac{1}{n} \\ &\mathrm{f}_3^{\max} = n \times n, \quad \mathrm{f}_3^{\min} = 0 \end{aligned} . \tag{12}$$

Extremenesses exist in the above function extrema. Extremum intervals of objective functions in the feasible region are included in the above estimative extremum intervals. The linear membership function only measures the degree of closeness, by which the selected solution approaches the ideal solution. Thus, the estimation does not affect the establishment of the membership functions and solution of the vector objective optimization under fuzzy method. Then new membership functions can be written as:

$$\begin{aligned} &\mu_1(\mathrm{f}_1(\mathbf{x})) = 1 - \tfrac{1}{m}\sum_{i=1}^{m}(\mathbf{A}_i\mathbf{x} - y_i)^2 \\ &\mu_2(\mathrm{f}_2(\mathbf{x})) = \left(\sum_{j=1}^{n} x_j \ln x_j\right) \Big/ \ln n \\ &\mu_3(\mathrm{f}_3(\mathbf{x})) = 1 - \frac{\mathbf{x}^{\mathrm{T}} \cdot \mathbf{S} \cdot \mathbf{x} + \mathbf{x}^{\mathrm{T}} \cdot \mathbf{x}}{2n^2} \end{aligned} . \tag{13}$$

A feasible fuzzy operator is selected to integrate several fuzzy specifications in one predetermined mode. The decision-maker can select and design integration mode to meet the requirement. The minimum operator[24] is selected as follows:

$$\mu_D(\mathbf{x}) = \min_{\mathbf{x}\in X}(\mu_1(\mathrm{f}_1(\mathbf{x})), \mu_2(\mathrm{f}_2(\mathbf{x})), \mu_3(f_3(\mathbf{x}))). \tag{14}$$

The Zimmermann algorithm model[24] was selected for two reasons: high computational efficiency and low complexity. The objective of this algorithm is to maximize the satisfactory index λ within the objective set and the solution $\mathbf{x}^*$ of the original problem. Its mathematical model is expressed as:

$$\begin{cases} \max \quad \lambda \\ \text{s.t.} \quad \lambda \le 1 - \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}(\mathbf{A}_i\mathbf{x} - y_i)^2 \\ \qquad \lambda \le \left(\displaystyle\sum_{j=1}^{n} x_j \ln x_j\right) \Big/ \ln n \\ \qquad \lambda \le 1 - \dfrac{\mathbf{x}^{\mathrm{T}} \cdot \mathbf{S} \cdot \mathbf{x} + \mathbf{x}^{\mathrm{T}} \cdot \mathbf{x}}{2n^2} \\ \qquad \mathrm{h}(\mathbf{x}) - \dfrac{m}{2} = 0 \\ \qquad \lambda \in [0, \quad 1] \end{cases} . \tag{15}$$

To solve Eq. (15), various solutions according to specific applications have been proposed.[26–31] The conventional methods for single objective optimization are not so feasible because of the following three reasons: (1) Zimmermann algorithm adopts uncompensated integration operator,[22] and increase of the grade of membership in the fuzzy intersection does not affect the whole grade of membership of the compositive fuzzy set. A Fuzzy algorithm based on it does not guarantee to converge to a satisfactory solution of the original problem. (2) The processing of nonlinear constraint condition in an optimization problem greatly increases difficulty. Consequently, the advantage of fuzzy vector objective optimization cannot be effective. (3) The image reconstruction by iterative method is time-consuming as the image vector is high dimensional, and the conventional optimization algorithm is incompetent.

We rewrite Eq. (21) as

$$\begin{cases} \max\ \lambda = \min \left\{ \begin{aligned} & 1 - \frac{1}{m}\sum_{i=1}^{m} (\mathbf{A}_i \mathbf{x} - y_i)^2, \\ & \left(\sum_{j=1}^{n} x_j \ln x_j \right) \Big/ \ln n, \\ & 1 - \frac{\mathbf{x}^{\mathrm{T}} \cdot \mathbf{S} \cdot \mathbf{x} + \mathbf{x}^{\mathrm{T}} \cdot \mathbf{x}}{2n^2} \end{aligned} \right\} \\ \text{s.t.}\ \ \mathrm{h}(\mathbf{x}) - \frac{m}{2} = 0 \end{cases}. \tag{16}$$

A summary of the FVOO method for image reconstruction from incomplete projections is described below:

(i) An initial image vector $\mathbf{x}^0$ is introduced by CBP from fan-scan projections. For incomplete projections, the initial image quality may not be perfect (see Fig. 2(a)). A termination error scalar $0 < \varepsilon << 1$ and two max satisfactory indices $\lambda = 0, \lambda' = 0$ are chosen. Let $k = 0$;

(ii) $\mu_1(\mathrm{f}_1(\mathbf{x}^k))$, $\mu_2(\mathrm{f}_2(\mathbf{x}^k))$, $\mu_3(\mathrm{f}_3(\mathbf{x}^k))$ are three different grades of membership. The minimum among them is selected to assign λ, $\lambda = \min(\mu_1(\mathrm{f}_1(\mathbf{x}^k)), \mu_2(\mathrm{f}_2(\mathbf{x}^k)), \mu_3(\mathrm{f}_3(\mathbf{x}^k)))$;

(iii) The minimum membership function and the constraint condition are constituted as a new optimization problem:

$$\begin{aligned} & \max\ \ \lambda = \mu_i(\mathrm{f}_i(\mathbf{x})) \\ & \text{s.t.}\ \ \mathrm{h}(\mathbf{x}) - \tfrac{m}{2} = 0 \end{aligned}. \tag{17}$$

It can be solved by iteration based on Kuhn-Tucker condition.[21] The solution is $\mathbf{x}^{k+1}$;

(iv) $\mu_1(\mathrm{f}_1(\mathbf{x}^{k+1}))$, $\mu_2(\mathrm{f}_2(\mathbf{x}^{k+1}))$, $\mu_3(\mathrm{f}_3(\mathbf{x}^{k+1}))$ are still three various grade of membership. Let $\lambda' = \min\{\mu_1(\mathrm{f}_1(\mathbf{x}^{k+1})), \mu_2(\mathrm{f}_2(\mathbf{x}^{k+1})), \mu_3(\mathrm{f}_3(\mathbf{x}^{k+1}))\}$;

(v) If $\lambda' < \lambda$, let $\mathbf{x}^* = \mathbf{x}^k$, output $\mathbf{x}^*$ and stop. Else go to (vi);

(vi) If $\lambda' - \lambda < \varepsilon$, let $\mathbf{x}^* = \mathbf{x}^{k+1}$, output $\mathbf{x}^*$ and stop. Else go to (vii);
(vii) Let $\lambda = \lambda'$, $k = k+1$, go to (iii).
Finally, $\mathbf{x}^k$ convergences the ideal solution $\mathbf{x}^*$.

5. Experiment Results

The application of FVOO method for image reconstruction have been tested by using two sets of different projections: (1) simulated projections with noise based on self-defined Sheep-Logan head model, which different from the normal Sheep-Logan head model, and contains 11 various shapes and densities ellipses. The elliptical phantom is contained in a rectangular region divided into 128×128 pixels (see Fig. 1). (2) projections from an industrial scanner.

The incomplete simulated projections generated from a fan scan consist of 180 equispaced projection views from 0° to 360°, with 128 centrosymmetric and equispaced rays per projection view. The projections with noise are generated as:

$$\begin{aligned} \hat{y}_i &= y_i + e_i \\ e_i &= N(0, 0.03^2) \cdot y_i \end{aligned}, \tag{18}$$

where y_i is accurate projection and $N(0, 0.03^2)$ represent a Gaussian probability distribution with zero mean and a variance $(0.03)^2$. The selection of the mean and variance is base upon comparison experiments between simulated projections and real projections.

For comparison, the reconstructed image from the same projections by Convolution Back Projection with Ram_Lak filter (CBP), General Algebraic Reconstruction Techniques (ART), Vector Objective Optimization (VOO)[20] and FVOO are shown in Fig. 2. The latter three images are obtained after several iterations.

In order to illustrate the solution process of FVOO image reconstruction, the diagram of iterative satisfactory index λ versus the iteration number is shown in Fig. 3. It is clear that satisfactory index λ increases with the iteration number, given an acceptable initial image vector $\mathbf{x}^0$. It shows integrative improvement of various objective functions.

Fig. 1. Original image of self-defined Sheep-Logan head model.

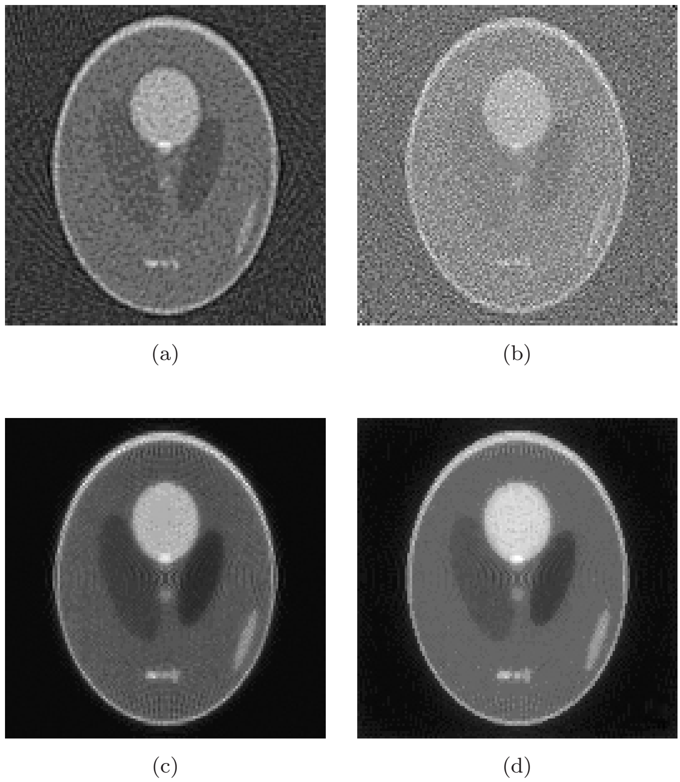


Fig. 2. Images reconstructed by (a) CBP (b) ART (c) VOO and (d) FVOO.

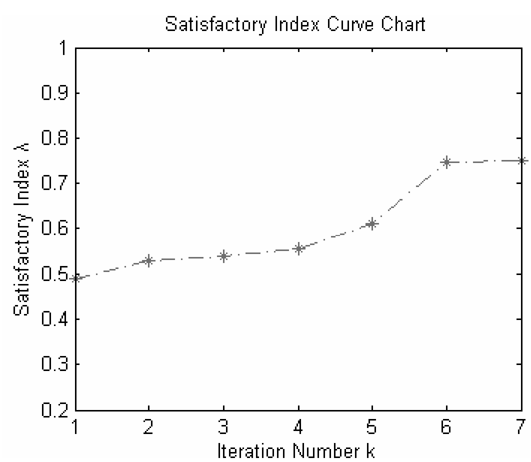


Fig. 3. Satisfactory index λ versus iteration number k in FVOO image reconstruction.

Table 1. Reconstructed error and time consumption.

Algorithm	Iteration number	Error	Iteration time (s)
CBP	–	0.593	1.03
ART	20	0.392	25.44
VOO	13	0.049	21.58
FVOO	7	0.018	49.58

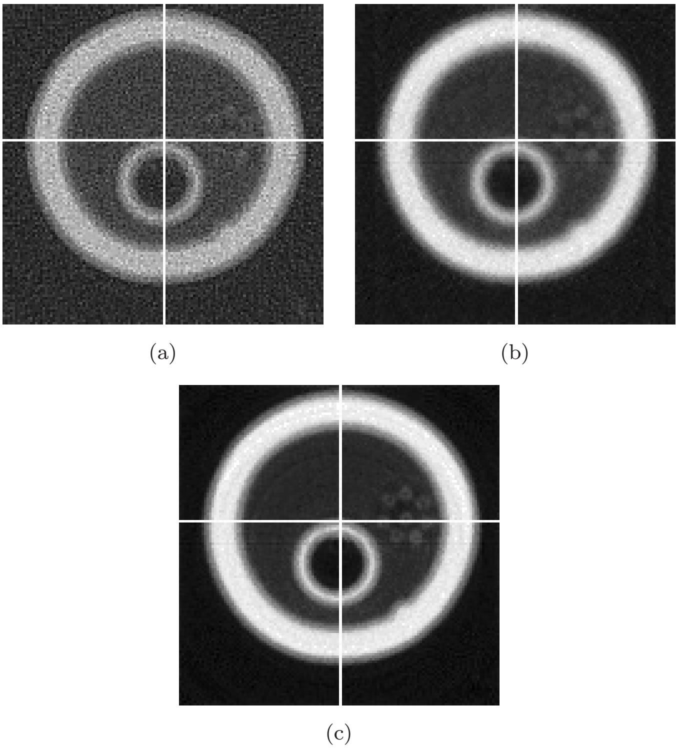
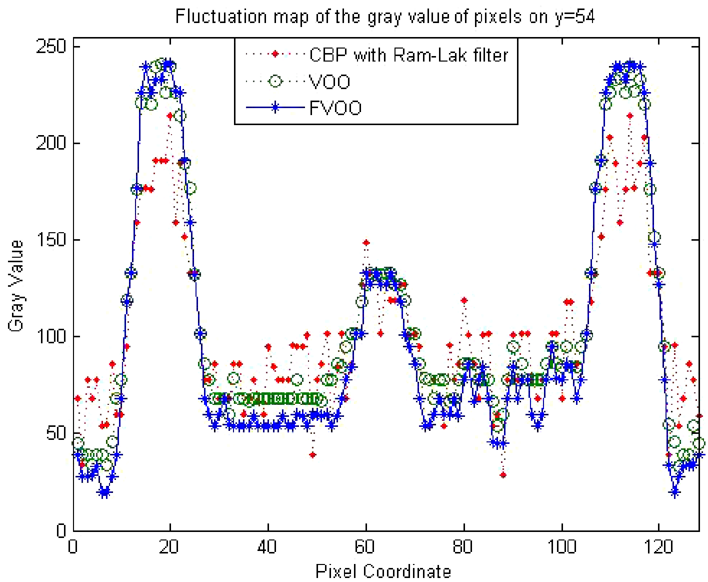


Fig. 4. Cross sectional images reconstructed from the real industrial scanner data by using (a) CBP with Ram-Lak filter (b) VOO (c) FVOO.

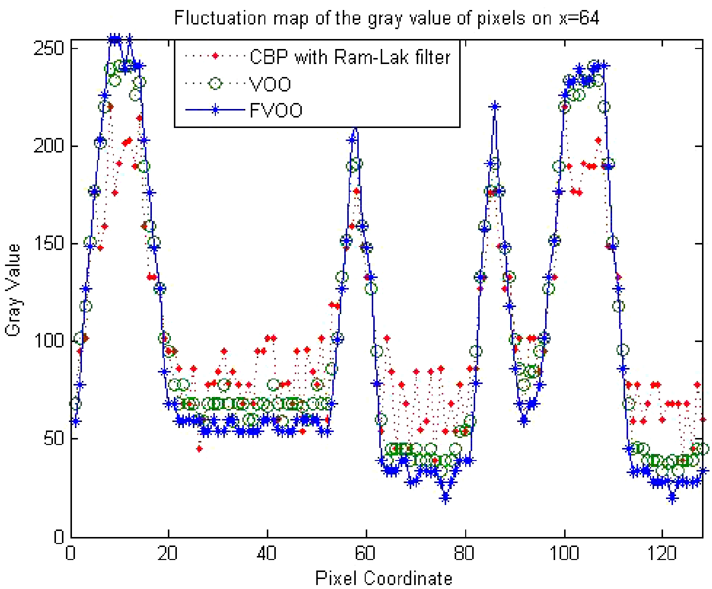
The reconstructed error is defined by Eq. (19) in the following:

$$e = \frac{\sum_j (x_j - \hat{x}_i)^2}{\sum_j (x_j)^2}, \tag{19}$$

where x_j is normal pixel value and $\hat{x}_i$ is reconstructed pixel value, and the computational time and the iteration number of different algorithms are shown in Table 1. The computational time of one image reconstruction method is different under different computer configure. So the iteration time in Table 1 is shown for intuitionistic comparison among four methods.



(a)



(b)

Fig. 5. Two profiles of the gray value of pixels on $y = 54$ and $x = 64$ corresponding to the images in Figs. 4(a), (b) and (c).

To further test the performance of the FVOO method, some experimental results are presented in Fig. 4, which were reconstructed from real industrial scanner data in fan scan by CBP with Ram-Lak filter, VOO and FVOO methods respectively. The projections are 180 equispaced projection views in the interval $[0, 2\pi]$, with 128 centrosymmetric and equispaced rays per projection view.

Furthermore, to intuitively compare the performance of above three methods in spatial resolution, the pixels on the straight lines $y = 54$ and $x = 64$, with y representing the number of rows and x representing the number of columns of the pixel matrix of the image, in Figs. 4(a), (b) and (c) are selected, which are highlighted by the bright lines in Fig. 4. The profiles along each of these bright lines are plotted in Fig. 5 respectively.

The above results and analysis can demonstrate the advantage of FVOO over other image reconstruction methods in terms of the reconstruction error, smoothness and gray value resolution, given the same noise and incomplete projections.

6. Conclusion

In this paper, a novel vector objective optimization method FVOO method is proposed for image reconstruction from incomplete projections. The method stresses multi-features satisfaction in reconstructed images, which is different from single objective optimization. It introduces membership function as an objective function or constraint function, and a minimum operator to integrate various membership functions, thereby establishing a new mathematical model for image reconstruction. To effectively solve the 'optimal' trade-off solution of the reconstruction model, an iterative min-max algorithm is presented, whose reconstructed image vector converges to the effective resolution. The performance of the FVOO algorithm has been tested by using simulated projections with noisy and real projections. The results have shown that this algorithm reduces the reconstructed error, suppresses noise and removes artifacts, for image reconstruction from incomplete projections.

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Appendix

The solution to the extremum of the objective function, however, increases computational complexity. Maxima and minima of the objective functions are estimated corresponding to Eq. (11):

(i) $f_1(\mathbf{x}) = ||\mathbf{Ax} - \mathbf{y}||^2$.

The projections $\mathbf{y}$ are normalized before the image reconstruction in order to control the result. Every vector y_i in the projection vector is normalized to $[0, 1]$. The processing does not influence the image reconstruction, and is easy to reconstruction. The pixel x_j is also normalized between 0 to 1. Consequently $\mathbf{A}_i\mathbf{x}$ and y_i are in the same order of magnitude. Thus we can deduce $(\mathbf{A}_i\mathbf{x} - y_i)^2 \in [0, 1]$, and $f_1(\mathbf{x}) \in [0, m/2]$. This can be expressed as

$$f_1^{\max} = \frac{m}{2}, \quad f_1^{\min} = 0, \tag{A2}$$

(ii) $f_2(\mathbf{x}) = \sum_{j=1}^{n} x_j \ln x_j$.

Based on the maximum discrete entropy theorem in information theory, we know that the value of $-\sum_{j=1}^{n} x_j \cdot \ln x_j$ is maximum when x_j is rectangular distribution.[32] It follows that $f_2^{\min}(\mathbf{x}) = \ln \frac{1}{n}$. Due to $x_j \in [0, 1]$, $x_j \cdot \ln x_j \leq 0$ and $\lim_{\substack{x_j \to 0 \\ x_j \to 1}} x_j \cdot \ln x_j = 0$, we can obtain the following equation:

$$f_2^{\max} = 0, \quad f_2^{\min} = \ln \frac{1}{n}. \tag{A3}$$

(iii) $f_3(\mathbf{x}) = \frac{1}{2}(\mathbf{x}^{\mathrm{T}}\mathbf{A} \cdot \mathbf{x})$

A conservative estimation is directly shown as

$$f_3^{\max} = n \times n, \quad f_3^{\min} = 0. \tag{A4}$$

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